Efficient Numerical Schemes for Solving Fractional Order Differential Equations

SYNOPSIS

SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENT OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

SUBMITTED BY

ATUL KUMAR

UNDER THE SUPERVISION OF

Dr. Madhukant Sharma
Co-Supervisor
DA-IICT Gandhinagar

Dr. Sumita Dahiya
Supervisor
Department of Mathematics

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE

DAYALBAGH EDUCATIONAL INSTITUTE (DEEMED UNIVERSITY)

DAYALBAGH, AGRA-282005

2022

1. Introduction

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders. In particular, this discipline involves the notion and methods of solving differential equations involving fractional derivatives of the unknown function. Fractional calculus is a natural extension of ordinary calculus, where integrals and derivatives are defined for arbitrary real orders. Since the 17th century, when fractional calculus was born, numerous different derivatives have been presented: Riemann-Liouville, Hadamard, Grunwald-Letnikov, and Caputo, just to mention some [17]. The subject of fractional calculus has applications in diverse and widespread fields of engineering and science. It has been used to model physical and engineering processes that are found to be best described by fractional differential equations. The fractional derivative models are used to accurately model those systems that require accurate damping modeling. In these fields, various analytical and numerical methods including their applications to new problems have been proposed in recent years. Numerous papers have appeared in which integer-order, standard differential equations, modeling processes of relaxation, diffusion, oscillation, and wave propagation are specified to fractional-order differential equations. The simple idea behind this is that the order of differentiation need not be an integer but a fractional number (i.e. d^{α}/dt^{α} with $0 < \alpha < 1$). Now a days, fractional variable order differential and integral operators, appear in mathematical models of many real-world phenomena namely ecological effects, blood flow issues, anomalous diffusion, spreading of diseases, control processes, and the rest. The variable-order fractional derivative, which are extension of constant-order fractional derivative, have been suggested and considered good mathematical operators to depict the memory property that changes with time or spatial co-ordinates. The variable order fractional operators are one of the fractional operators which are dependent on their non-stationary power-law kernel, can define the memory and hereditary properties of many physical phenomena and processes. And so, they accurately characterize complex physical systems and processes [6]. In general, the numerical methods are engaged as well-organized developed methods for the numerical approximation of variable order fractional differential equations [24, 21]. Numerical integration is a basic tool for obtaining the approximate value of the definite integrals where the analytical integrations are hard to evaluate. Numerical integrations for the

fractional integrals are also important in developing the algorithms for solving applied problems defined by fractional derivatives. Generally, most of the fractional differential equations do not have exact solutions. Particularly, there is no known method to solve fractional boundary value problems exactly because of the extensive applications of fractional differential equations. This synopsis "Efficient numerical scheme for solving Fractional Order Differential Equations" is based on the definitions of fractional derivatives and integration, and also on special functions as Gamma function, Beta function, Mittag-Leffler Function, and Miller-Ross function.

2. Definition

2.1. Definition [Riemann-Liouville Fractional Integral]

Let [a b] ($-\infty < a < b < \infty$) be a finite interval on the real-axis R. The Riemann-Liouville fractional integrals (left-sided and right-sided) [11] of order α, with $\alpha > 0$, are defined by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \; ; \quad x > a \qquad (1)$$

$$\text{and } I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt \; ; x < b$$

2.2. Definition [Riemann-Liouville Derivative]

Riemann-Liouville fractional derivatives (left-sided and right-sided) [11] of function f of order α are given by

$$D_{a+}^{\alpha} f(x) = \frac{d^n}{dx^n} I_{a+}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)dt}{(x-t)^{\alpha+1-n}} \qquad (2)$$

$$\text{and } D_{b-}^{\alpha} f(x) = (-1)^n \frac{d^n}{dx^n} I_{b-}^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{f(t)dt}{(t-x)^{\alpha+1-n}}$$

Where $n - 1 < \alpha < n$

2.3. Definition [Hilfer Derivative]

Hilfer fractional derivatives (left-sided and right-sided) [11] of function f of order $n - 1 < \alpha < n$ and type $0 \le \beta \le 1$ defined by

$$D_{a+}^{\alpha,\beta} f(x) = I_{a+}^{\mu-\alpha} \frac{d^n}{dx^n} I_{a+}^{(n-\alpha)(1-\beta)} f(x) \qquad (3)$$

$$\text{and } D_{b-}^{\alpha,\beta} f(x) = I_{a+}^{\mu-\alpha} \left(-\frac{d}{dx}\right)^n I_{b-}^{(n-\alpha)(1-\beta)} f(x)$$

2.4. Definition [Caputo Derivative]

Caputo derivatives (left-sided and right-sided) [11] of function f of order α are given by

$$c_{D}{}^{\alpha}_{a+}\mathrm{f(x)} = \frac{1}{\mathbb{\Gamma}(n-\alpha)} \int_{a}^{x} \frac{f^{n}(t)dt}{(x-t)^{\alpha+1-n}} \qquad (4)$$

$$\text{and } c_{D}{}^{\alpha}_{b-}\mathrm{f(x)} = \frac{(-1)^{n}}{\mathbb{\Gamma}(n-\alpha)} \int_{x}^{b} \frac{f^{n}(t)dt}{(t-x)^{\alpha+1-n}}$$

where n - 1 < α < n

2.5. Definition [Nutting's Law]

This relation is frequently called Nutting's law $\sigma(t) = \upsilon D^{k}\varepsilon(t)$ where 0< k< 1, ε(t) is strain $\sigma(t)$ is stress and υ is material constant [8].

3. Literature Review

Lorenzo et al. [5] presented fractional calculus, containing differentiation and integration had a history of more than three hundred years. Sun et al. [8] aimed to provide a survey of the recent relevant literature and findings in primary definitions, models, numerical methods and their applications for fractional derivatives. That review first offered an overview over the existing definitions proposed from different physical and application backgrounds, and then reviewed several widely used numerical schemes in simulation. They [2,3,4,9,10,12,15,16,18,19,23,24] proposed and summarized the work of variable order fractional differential equations with the help of different fractional derivatives and different numerical techniques and proved the existence of the solutions of such variable order fractional differential equations. Sun et al. [7] worked for constant order and variable order fractional differential equations, and they showed that characterizing the memory property of systems was a challenging issue in the modeling and analysis of complex systems. In the following, all the references are related to (constant) order fractional differential equations. Atangana et al. [1] proposed a new fractional derivative with non- local and non-singular kernel and presented some useful properties of new derivative and applied it for solving the fractional heat transfer model. Toufik et al. [14] introduced the new concept of fractional differentiation with non-local and non-singular kernel, in order to extend the limitations of the conventional Riemann-Liouville and Caputo fractional derivatives. The numerical scheme was applied

for solving linear and non-linear fractional differential equations. Jain [20] aimed to use the alternative numerical scheme given by Gnitchogna and Atangana to solve partial differential equation with integer and non-integer differential operator. He applied this method to fractional diffusion model and fractional Buckmaster models with non-local fading memory. Owolabi et al.

[13] modeled the range of new fractional-order love dynamics using the Caputo (power-law), Caputo–Fabrizio and Atangana–Baleanu fractional operators. It was observed that modelling of interpersonal and romantic love affairs with fractional derivative could exhibit some strange emotional attractors. Ali et al. [22] showed fractional derivative was non local, which was more suitable to simulate physical phenomena and provided more accurate models of physical systems such as earthquake vibration and polymers. It suggested a new numerical approach for the fractional diffusion-wave equation. The fractional-order derivative was in the Riemann-Liouville sense.

4. Objective

- Prime objective is to derive new numerical schemes for solving fractional differential equations of second and higher order.
- To study the error analysis of the developed schemes.
- Applications of developed numerical schemes on fractional diffusion model.
- To minimize computational efforts and maximize accuracy of the numerical solutions obtained.

5. Methodology:

We have numerous methods for solving fractional differential equations. Some of these have been presented for obtaining the exact solutions and approximate analytic solutions of fractional differential equations. Some of these methods use integral transform in order to reduce equations into simpler equations, and some of these methods provide solutions in series form those converge to the exact solution. Fractional differential equations can be solved by using such kinds of methods.

Some more methods used are Fourier Transforms, Eigen vector expansion, Laplace transforms, Adomian decomposition method, variational iteration method, finite difference method, the power series method, fractional differential transform method, and homotopy analysis method.

For example, we have General problem of Caputo fractional derivative

$${}_0^C D_t^{\alpha} y(t) = f(t,y(t));\ t > 0,\ 0 < \alpha \leq 1 \text{ and } y(0) = y_0, \text{ then we have}$$

$$y(t) = y_0 + \frac{1}{\mathbb{\Gamma}(\alpha)} \int_0^t f(\tau, y(\tau))\ (t-\tau)^{\alpha-1}\ d\tau \text{ where } 0 \leq t \leq b$$

Many methods are already developed but we are planning (intending) to solve these types of problems by methods such as finite difference method, interpolation etc. There are numerous approaches to the generalization of the idea of differentiation to fractional orders, e.g., Riemann-Liouville, Hilfer, Caputo and generalized function approach. The Riemann-Liouville fractional derivative is typically used but this approach is not very appropriate for real world problems. Caputo presented alternate definition which has the advantage of defining integer order initial condition for fractional order differential equations unlike the Riemann-Liouville approach which develops its definition from repeated integration. This approach is typically used in numerical algorithms. In our work, we intend to use some such appropriate approaches. We also intend to use some softwares such as MATLAB/MATHEMATICA/MAPLE in solving equations and plotting graphs.

6. Limitations:

Fractional calculus delivers a virtuous tool to describe physical memory and heredity. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various material and processes. This is the main advantage of fractional derivatives in comparison with classical derivatives. The disadvantage is the complexity to find the analytic solutions of differential equations involving fractional derivatives. In our problem, we will use MATLAB as a platform to handle such complicated problems numerically. Note that there are only limited tool boxes available in MATLAB such as FOMCON, etc. for solving fractional differential equations. Fractional differential equations are without a doubt, notion of a higher complication by nature but once learnt, they can be solved.

References

[1]. A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. Thermal Science. 20, Issue 2 (2016), 763-769.

[2]. A. H. Bhrawy, M. A. Zaky, Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation. Nonlinear Dynamics. 80, No 1-2 (2015), 101–116.

[3]. A. Razminia, A. F. Dizaji, V. J. Majd, Solution existence for non-autonomous variable-order fractional differential equations. Mathematical and Computer Modelling. 55, No 3-4 (2012), 1106–1117.

[4]. C. F. Lorenzo, T. T. Hartley, Initialization, conceptualization, and application in the generalized fractional calculus. Critical Reviews in Biomedical Engineering. 35, No 6 (2007), 477–553.

[5]. C. F. Lorenzo, T. T. Hartley, Variable order and distributed order fractional operators. Nonlinear Dynamics. 29, No 1 (2002), 57–98.

[6]. G. Diaz, C. F. M. Coimbra, Dynamics and control of nonlinear variable order oscillators. Nonlinear Dynamics (Ed. T. Evans), InTech (2010), 129–144.

[7]. H. G. Sun, W. Chen, H. Wei1, and Y.Q. Chen, A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems. The European Physical Journal Special Topics. 193, (2011), 185–192.

[8]. H. Sun, A. Chang, Y. Zhang, W. Chen, A Review on variable-order fractional differential equations: Mathematical foundation, physical model, numerical methods and applications. Fractional Calculus and Applied Analysis. 22, Issue 1 (2019), 27–59.

[9]. H. Zhang, F. Liu, M. S. Phanikumar, M. M. Meerschaert, A novel numerical method for the time variable fractional order mobile-immobile advection-dispersion model. Computers & Mathematics with Applications. 66, Issue 5 (2013), 693-701.

[10]. J. Cao, Y. Qiu, A high order numerical scheme for variable order fractional ordinary differential equation. Applied Mathematics Letters. 61(2016), 88–94.

[11]. J. Vanterler da C. Sousa and E. Capelas de Oliveira, on the ψ -Hilfer fractional derivative. Classical Analysis and ODEs. (2017), 1-28.

[12]. K. Kumar, R. K. Pandey, S. Sharma. Approximations of fractional integrals and Caputo derivatives with application in solving Abel's integral equations, Journal of King Saud University - Science, 31, Issue 4,(2019), 692-700.

[13]. K. M. Owolabi, Mathematical modelling and analysis of love dynamics: A fractional approach. Physica A: Statistical Mechanics and its Applications, 525 (2019), 849-865.

[14]. M. Toufik, A. Atangana, New numerical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models. The European Physical Journal Plus. 132, Issue 10 (2017), 1-16.

[15]. M. Zayernouri, G. E. Karniadakis, Fractional spectral collocation methods for linear and nonlinear variable order FPDEs. Journal of Computational Physics. 293 (2015), 312–338.

[16]. N. H. Sweilam, S. M. Al-Mekhlafi, Numerical study for multi-strain tuberculosis (TB) model of variable-order fractional derivatives. Journal of Advanced Research. 7, No 2 (2016), 271–283.

[17]. N. I. Mahmudov, M. Awadalla and K. Abuassba, Nonlinear sequential fractional differential equations with nonlocal boundary conditions. 2017. Advances in Difference Equations. 2017, No. 319 (2017), 1-15.

[18]. R. Lin, F. Liu, V. Anh, I. Turner, Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. Applied Mathematics and Computation. 212, No 2 (2009), 435–445.

[19]. S. G. Samko, B. Ross, Integration and differentiation to a variable fractional order. Integral Transforms and Special Functions.1, Issue 4 (1993), 277–300.

[20]. S. Jain, Numerical analysis for the fractional diffusion and fractional Buckmaster equation by the two-step Laplace Adam-Bashforth method. The European Physical Journal Plus. 133, Article number: 19 (2018), 1-11.

[21]. S. Shen, F. Liu, J. Chen, I. Turner, V. Anh, Numerical techniques for the variable order time-fractional diffusion equation. Applied Mathematics and Computation. 218, No 22 (2011), 10861–10870.

[22]. U. Ali, M. A. Khan, M. M. A. Khater, A. A. Mousa, R. A. M. Attia. A New Numerical Approach for Solving 1D Fractional Diffusion-Wave Equation. Journal of function spaces. 2021, Article ID 6638597, 2021, 1-7.

[23]. X. Zhao, Z. Sun, G.E. Karniadakis, Second-order approximations for variable order fractional derivatives: Algorithms and applications. Journal of Computational Physics. 293 (2015), 184–200.

[24]. Y. Chen, Y. Wei, D. Liu, H. Yu, Numerical solution for a class of nonlinear variable-order fractional differential equations with Legendre wavelets. Applied Mathematics Letters. 46 (2015), 83–88.